Journal of Statistical Physics, Vol. 62, Nos. 1/2, 1991

On Correction to Scaling for Two- and Three-Dimensional Scalar and Vector Percolation

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Received August 1, 1990

A reanalysis of the resistance R of two-and three-dimensional superconducting networks at the percolation threshold p_c, together with the previous results for the elastic moduli K of such networks, shows that there is a *unified* description of finite-size scaling for scalar and vector transport properties of percolation systems. For a network of linear size L at p_c, and for *both* scalar and vector percolation in *both* two and three dimensions, K and R scale with L as $L^{-x}[a_1 + a_2(\ln L)^{-1} + a_3 L^{-1}]$, where x is the ratio of the associated critical exponent of K or R and the correlation length exponent ν of percolation. Although our estimates of x for the resistance of percolation networks are consistent with the previous results, they do indicate that in *both* two and three dimensions and for *both* scalar and vector percolation, the leading nonanalytic correction-to-scaling exponent is *zero*. From a reanalysis of data on diffusion on percolation clusters at p_c, we propose that such correction-to-scaling terms are a general property of *dynamics* of percolation clusters. We also suggest that for two-dimensional percolation the conductivity exponent t and the superconductivity exponent s are given by $s = t = \nu - \beta/4 = 187/144 = 1.2986...$, and the elasticity exponent f is given by $f = t + 2\nu = 571/144 = 3.9652...$, where β is the exponent of the strength of the infinite percolation cluster.

KEY WORDS: Percolation; finite-size scaling,; correction to scaling; conductivity; elasticity.

For the past two decades random percolation networks[1] have been an important tool for investigation of transport processes in disordered systems, such as porous media, polymer gels, and composite solids. A lot

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0022-4715/91/0100-0453$06.50/0

of the attention has been focused on the properties of such networks near the percolation threshold p_c of the network. Consider a percolation network in which a fraction p of the bonds are ordinary conductors (with finite conductance or resistance) and the rest are insulators. As p approaches p_c from above, the bulk dc conductivity G of the network vanishes according to the power law $G \sim \varepsilon^t$, whereas the conductivity of a network in which a fraction p of the bonds are superconductors (with zero resistance) and the rest are ordinary conductors diverges as p_c is approached from below according to the scaling law $G \sim \varepsilon^{-s}$, where $\varepsilon = |p - p_c|$. Similarly, in a network of linear elastic elements (e.g., linear springs) which can be stretched and/or bent, and in which a fraction p of the springs have a finite elastic constant e and the rest have a zero elastic constant, the elastic moduli K of the network vanish as p approaches p_c as $K \sim \varepsilon^f$, whereas if $e = \infty$ for a fraction p of the springs (i.e., the springs are totally rigid) and e is finite for the rest of the springs, one has $K \sim \varepsilon^{-\tau}$. Moreover, near p_c the correlation length ξ diverges as $\xi \sim \varepsilon^{-\nu}$, and the strength $P(p)$ of the network, i.e., the fraction of bounds or sites in the sample-spanning (infinite) cluster, vanishes as $P(p) \sim \varepsilon^\beta$. In two dimensions (2d) a duality argument establishes rigorously[2,3] that $s = t$. It has also been suggested[4,5] that $f = t + 2\nu$ and[6] $\tau = \nu - \beta/2$. It is currently an open question whether t and s are related to ν, β, and other percolation exponents that characterize geometrical properties of percolation clusters. These exponents have been estimated by a variety of methods. In Table I we present the currently accepted values of these exponents.

Among various methods that have been used to estimate these exponents, the transfer matrix algorithm,[7–9] when combined with finite-size scaling analysis[10] (FSSA), seems to be the most accurate method of estimating t, s, f, and τ. One constructs very long bars of random resistors or springs[11] and calculates its resistance (conductance) or elastic moduli during the construction. The associated critical exponent is then obtained through FSSA by extrapolating from bars of different lengths or cross sections. According to FSSA, for a network of size $L \times L \times L$ (or $L \times L$ in 2d)

Table I. Currently Accepted Values of the Critical Exponents in d Dimensions

d	ν	β	t/ν	s/ν	f/ν	τ/ν
1	1	0	0	1	0	1
2	4/3	5/36	0.9745 ± 0.0015	0.9745 ± 0.0015	2.97 ± 0.03	0.92 ± 0.03
3	0.88	0.41	2.27 ± 0.20	0.835 ± 0.005	4.30 ± 0.10	0.74 ± 0.04

or a bar of size $L \times L \times n$ (or $L \times n$ in 2d), where n is very large (or order of 10^7–10^9), at p_c one has

$$K \sim L^{-x}[a_1 + a_2 h_1(L) + a_3 h_3(L)] \tag{1}$$

where $x = f/\nu$, and the resistance R scales as

$$R \sim L^{y}[b_1 + b_2 g_1(L) + b_3 g_2(L)] \tag{2}$$

where $y = t/\nu$ or $-s/\nu$. Here the functions g_1, g_2, h_1, and h_2 represent the leading correction-to-scaling (CTS) terms which are particularly important for small to moderate values of L, and a's and b's are numerical constants. Using this technique and the special-purpose computer PERCOLA,[12] Normand *et al.*[13] and Normand and Herrmann[14] (NHH) studied resistances of bond and site percolation networks, and obtained very precise estimates of s in both 2d and 3d (see Table I).

The goals of this paper are as follows. (1) We reanalyse the data of NHH to see whether it is possible to obtain more precise estimates of s. (2) We show that for *both* R and K, $g_1 = h_1 = (\ln L)^{-1}$ and $g_2 = h_2 = L^{-1}$ yield the most precise values of the exponents, by which we mean an estimate with the smallest estimated errors, *and* also the smallest sum of the squared deviations (SSD) between the predictions of K or G and the actual transfer matrix data. In addition, we show that this choice of g_1 and g_2 provides consistent estimates of s for *both* site and bond percolation data (as required by universality), i.e., using the proposed g_1 and g_2 yields estimates of s for site and bond percolation that, within their error bars, are consistant with one another, whereas in order to be consistent with the principle of universality, NHH had to *impose* certain choice to obtain *compromise* estimates of s. The proposed g_1 and g_2 imply that the leading nonanalytic CTS exponent is *zero*, while that of analytic CTS is *one* in *both* 2*d* and 3*d*. As such, these exponents appear to be *superuniversal*, i.e., independent of dimension. (3) Based on these results, we suggest scaling relations for t and s for low-dimensional systems, and provide some theoretical arguments in their support.

In the analysis of their data, NHH assumed that

$$R \sim L^{-y}(c_1 + c_2 L^{-\omega}) \tag{3}$$

where $y = s/\nu$. Here ω is the first universal CTS exponent. They suggested, based on the analysis of their data, that $\omega \simeq 1.2$ and 1.8 in 2d and 3d, respectively. However, if one uses (3) for the bond and site percolation data separately, one obtains estimates of y and ω that are *not* consistent with one another (within their estimated errors) and therefore the proposed values of ω [imposed on Eq. (3)] represent compromise values in order to obtain

universal estimates of y for both bond and site percolation data. On the other hand, we have already shown that[15] in estimating f for 3d systems and[6] τ for *both* 2d and 3d networks the functions

$$h_1(L) = (\ln L)^{-1} \tag{4}$$

$$h_2(L) = L^{-1} \tag{5}$$

provide the most accurate fits to the data. In our analyses, we have also used an equation similar to (3) and found that the quality of the resulting fits was worse than those provided by Eqs. (2), (4), and (5). Zabolitzky *et al.*[11] had already reached the same conclusion for estimating f for 2d systems. We now show that the same is also true of the data of NHH for R for *both* 2d and 3d networks.

Figure 1 presents the fits of the 2d data, for both bond and site percolation, using Eq. (2) together with (4) and (5). It is clear that the first are excellent and we obtain

$$s/\nu = 0.9748 \pm 0.0010 \tag{6}$$

This is slightly larger than that proposed by NHH (see Table I), while the estimated error is somewhat smaller. Our estimated errors are mainly due to the fluctuations in the data themselves, and the error due to fitting is negligible. To estimate these errors, we let each data point take on its

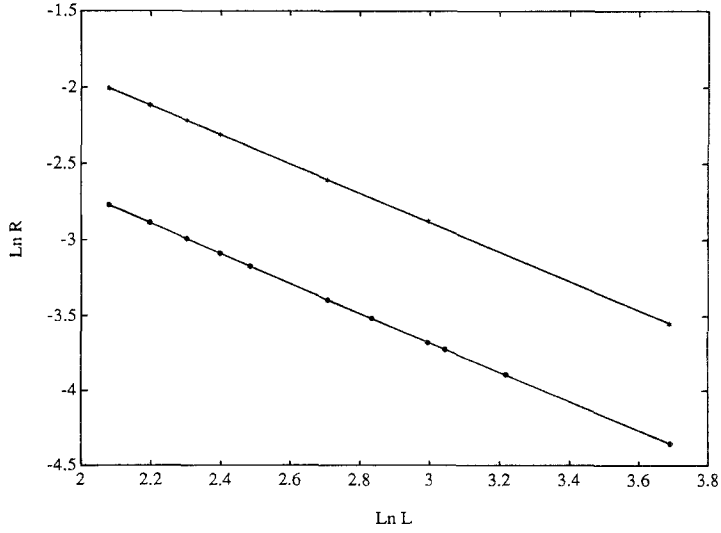

Fig. 1. Plot of ln R versus ln L for 2d site (*) and bond (•) percolation.

largest or smallest allowed value (according to its individual error as reported by NHH) and estimate the value of the exponent. The deviations between such estimates and the central value of the exponent given above yield the estimated errors. This method, we believe, provides a conservative estimate of the errors. A few points are worth mentioning here. First, as was also done by NHH, we analyzed the 2d data for $L \geqslant 7$, since the data for smaller values of L, especially those for bond percolation, showed significant small size dependence. However, since we have used two correction terms, the result using the data for *all* L's would be only slightly larger then Eq. (6) (we would obtain $s/\nu = 0.9753$), whereas using Eq. (3) does not provide such consistency in the results. Second, the SSD between the predictions, using Eq. (6), and the data is more than one order of magnitude smaller than that provided by Eq. (3). For example, for 2d site percolation, the SSD with Eqs. (2), (4), and (5) is about 10^{-5}, whereas it is only about 10^{-2} if one uses Eq. (3). Third, one can more generally use equations such as $h_1 = (\ln L)^{-\omega_1}$ and $h_2 = L^{-\omega_2}$, where ω_1 and ω_2 are two universal exponents. We have used such equations and have found that the quality of the resulting fits is always worse than that provided by Eqs. (4) and (5).

The difference between our 3d estimate of s/ν and that proposed by Normand and Herrmann (Table I) is somewhat larger than the corresponding value in 2d, though still within the combined error bars. Figure 2 presents the fits using Eqs. (2), (4), and (5), and it is clear that the fits are again excellent. We estimate that

$$s/\nu = 0.827 \pm 0.006 \tag{7}$$

As in the case of 2d systems, Eqs. (2), (4), and (5) provide consistent fits of bond and site percolation data, whereas (3) would not and one has to use compromise values of y and ω. This fit with (3) is, however, done at the expense of increasing the SSD between the predictions and the data by at least one or two orders of magnitude.

Although the difference between our estimates for the leading exponent y and those proposed by NHH is not very large, and all estimates are consistent with one another to within their estimated errors, our analysis provides a *unified* and *simpler* description of FSSA of scalar and vector percolation in *both* 2d and 3d. One no longer needs to estimate ω, and the CTS exponents are *a priori* known. Moreover, our estimates of s are obtained with much smaller SSD between the predictions and the data and therefore from a statistical point of view are perhaps more reliable. Thus, in a sense they confirm NHH's estimates for the leading exponents. This, we believe, represents the main contribution of our

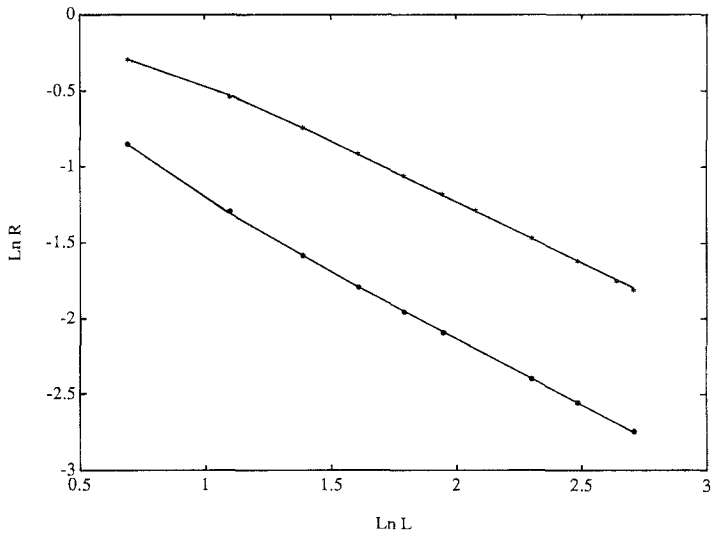

Fig. 2. Same as in Fig. 1, but for 3d percolation.

analysis. We do not, however, have a theoretical explanation for Eqs. (4) and (5).

To test whether such CTS terms are suitable for describing any other *dynamic* property of percolation clusters, we also reanalyzed the diffusion data of Roman.[16] Diffusion on percolation clusters at p_c is anomalous, and the root-mean-squared displacement r of a diffusing particle in such a system increases with time θ as $r \sim \theta^k$. Roman[16] has recently obtained accurate data for r as a function of θ for a simple-cubic network at p_c. We found that $r \sim \theta^k[d_1 + d_2(\ln \theta)^{-1} + d_3\theta^{-1}]$ provides a very precise estimate of k, and obtained $k \simeq 0.195 \pm 0.001$, which should be compared with 0.190 ± 0.003 that was suggested by Roman,[16] with our SSD being of the order of 10^{-5}. Our estimate of k is closer to series expansion estimate,[17] $k \simeq 0.198 \pm 0.003$, than Roman's, and is also close to the most recent Monte Carlo estimate,[18] $k \simeq 0.200$. On the other hand, if we reanalyze the more recent data of Paetzold,[19] who studied diffusion of interacting particles on 3d percolation clusters at p_c, we obtain $k \simeq 0.200 \pm 0.005$, which is again compatible with the series estimate,[17] Monte Carlo estimate,[18] and ours, but is *barely* consistent with Paetzold's, $k \simeq 0.183 \pm 0.010$. Based on these analyses, we suggest that our CTS terms are perhaps general characteristics of *dynamic* properties of percolation clusters.

Because of the high accuracy of the estimated values of t and s for 2d systems, we would like to speculate on the relation between them and

the geometrical exponents of percolation. For 2d systems, Alexander and Orbach[20] and Kertész[21] conjectured that $s = t = \nu - \beta/2 = 91/72 = 1.2639...$, which underestimates t and s, while Levinshtein *et al.*,[22] Starley,[23] and Aharony and Stauffer[24] proposed that $s = t = \nu = 4/3 = 1.333...$, which overestimates these exponents. Taking a simple arithmetic average of the conjectured values yield $s(d=2) = t(d=2) = \nu - \beta/4 = 187/144 = 1.2986$ (or $s/\nu = t/\nu = 187/192 = 0.9739...$), which is completely consistent with the data. Noting also that $s(d=1) = 1$, we suggest that

$$s = \nu - \beta/4 \tag{8}$$

for $d \leqslant d_l$, where d_l is some kind of lower critical dimensionality, the significance of which is discussed below. Since $t(d=1) = 0$, we also suggest that

$$t = (d-1)\nu - \beta/4 \tag{9}$$

for $d \leqslant d_l$. For $d \geqslant d_l$ the relations between s and t and the geometrical exponents of percolation will be discussed in a future paper. Equation (9), together with $f = t + 2\nu$, would then predict that $f(d=2) = 571/144 = 3.9652...$. We note that our conjecture for s and t for $d=2$ is consistent with a hypothesis of Pearson[25] that if the critical exponents of any 2d system are rational number, then the denominator of the exponents has to be of form $2^{n_1} \times 3^{n_2} \times 5^{n_3}$, where n_1, n_2, and n_3 are small integers or zero (in our case, $n_1 = 4$, $n_2 = 2$, and $n_3 = 0$). We also note that Eq. (9) is in agreement with the $(1+\varepsilon)$ expansion of t,[26] where $\varepsilon = d - 1$, if one uses the $(1+\varepsilon)$ expansions of ν and β due to Stephen.[27]

What is the significance of d_l? As one of us stated several years ago,[28] there is good reason to believe that there is a lower critical dimensionality d_l such that the relations between s and t and the geometrical exponents of percolation, if they do exist, would be different for d above or below d_l. Fucito and Parisi[29] and Harris and Lubensky[30] have shown that the ε-expansions for percolation critical exponents (here, $\varepsilon = 6 - d$) calculated from a φ^3 theory may break down at some *anomalous* dimensionality d_l, which they estimated to be close to three. They have argued that at $d = d_l$ the fourth-order potential in the field-theoretic formulation of the problem becomes relevant. If this is true, one may have a scaling relation between, e.g., t and the geometrical exponents of percolation which may yield an ε-expansion for t which is consistent with that derived directly,[31] and yet this scaling relation could be wrong at $d=2$, if d_l does actually exist and is larger than two. Therefore, we suggest that Eqs. (8) and (9) are exact for any d below d_l. The relations between s and t and geometrical exponents of percolation for d above d_l will be discussed in a future paper.[32]

Finally, we would like to suggest that one should perhaps reanalyze the existing series expansions for the conductivity exponent t and include CTS terms similar to what we suggest here. Currently, the series expansions yield[17,33] $t = 1.26$, in agreement with the Alexander–Orbach–Kertész conjecture for 2d percolation. Thus, it may be possible that by including the CTS terms that we are proposing here, one can resolve the discrepancy between the series estimate and the simulation data.

ACKNOWLEDGMENTS

We are grateful to H. J. Herrmann and C. J. Lobb for making available to us their data prior to publication, and to Dietrich Stauffer for many useful discussions and suggestions, encouragement, and a critical reading of the paper. One of us (M.S.) would like to thank the Supercomputer Center HLRZ-KFA for warm hospitality. This work was supported in part by the National Science Foundation grant CST8615160.

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Communicated by D. Stauffer